Mechanics of Vertical Moving Fluidized Systems with Mixed Particle Sizes

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Natural and industrial processes involve systems of nonuniform sized particles. Systematic study and analysis of such systems or of analogous mixtures are needed to improve design and control of these systems. The present work deals with mixed sized particles in a fluidized continuous system and is an extension of previous investigations in this laboratory on fluidized continuous systems of single sized particles (*1* to *3*) and batch fluidization of mixed sized particles (*4*). These studies illustrated the validity of the unique characteristic holdup-slip velocity relationship formulated by Lapidus and Elgin (*5*).

Extensive experimental (*1, 6* to *9*) and theoretical studies (*10* to *15*) were carried out to establish a quantitative relationship between holdup and slip velocity in various flow regimes. The quantitative relationship demonstrated in its general form was

$$V_s = V_{so} \cdot \epsilon \cdot f(1 - \epsilon) \tag{1}$$

where the function $f(1 - \epsilon)$ was found in the exponential form (*13, 15, 16*) or as a voidage ϵ raised to a power (*7, 9, 12*).

$$V_s = V_{so} \cdot \epsilon^n \tag{2}$$

The value of the exponent n was determined theoretically (*12*) or experimentally (*7, 9*). Few studies attempted to use the slip velocity-holdup relationship in mixed size systems. These attempts were limited to a definition of a mean diameter (*17*), or a mean terminal velocity of mixtures (*18*), expressed as

$$V_{Tm} = \frac{\Sigma X_i d^3_i}{\Sigma \dfrac{X_i d^3_i}{V_{Ti}}} \tag{3}$$

Such mean characteristics could be applied to uniform mixtures only but not to batch fluidized beds where segregation was always reported (*9, 19, 20*). In sedimentation, segregation was observed in dilute suspensions (*21*) and in more concentrated slowly settling systems (*22*).

Wen and Yu (*23*) stated that segregation occurred only for size ratios higher than 1.3. Hoffman et al. (*4*) fluidized binary and ternary mixtures of glass beads. He observed sharp size separation in mixtures of size ratios from 1.58 to 2.23, and partial separation in mixtures of size ratio 1.24. However, the total holdup of any mixture $(1 - \epsilon_T)$ calculated as

$$(1 - \epsilon_T) = \frac{1}{\Sigma \dfrac{X_i}{(1 - \epsilon_i)}} \tag{4}$$

showed excellent agreement with the experimental results. Similarly McCune and Wilhelm (*19*) measured exit concentrations and pressure drops in batch-fluidized binary mixtures and found the experimental results in agreement with values computed as if the total bed were composed of single sized segments in series.

Scarlett and Blogg (*20*) sampled along their batch-fluidized bed of nonuniform glass beads (70 to 190 μ). The authors presented profiles of size distributions which narrowed at any level as the water-fluidizing velocity increased. The size distribution at the lowest level did not vary with the fluid velocity, an effect which was attributed to mixing at the bottom of the bed.

Pruden and Epstein (*24*) analyzed the phenomenon of size segregation in a particulate fluidized bed. The analysis was based upon the assumption that the driving force to segregate was expressed by the difference in bulk density of the individual particle beds. The final expression derived for a binary mixture took the form of an index of stratification.

$$\frac{\rho_{B1} - \rho_{B2}}{\rho_d - \rho_c} = \epsilon_1 \left[\left(\frac{d_1}{d_2} \right)^{\frac{3-k}{k(n+1)}} - 1 \right] \tag{5}$$

The derived Equation (5) signifies stratification for any nonmonosized bed and predicts a continuously progressing process. In practice, however, the particles circulation (*2*) and hydrodynamic instabilities oppose stratification.

The experimental studies herein surveyed were supposed to deal with a steady state batch fluidization. In fact, batch fluidization may be carried out for an indefinite time, in contrast to sedimentation which is limited by its container depth. The time limit imposed on settling may inhibit measurable segregation. In fact, uniform settling was sometimes observed (*21, 22*). These observations suggested that continuously fluidized mixtures in short beds would not segregate.

ANALYSIS OF A FLUIDIZING MIXTURE

Dynamic stratification has not been investigated systematically, but obviously stratification is inversely proportional to the duration of flow and hence minimized in short beds.

The present work falls within the last category. An analysis based on the following assumptions is applied:

1. Absence of segregation of feeds, radial or longitudinal (stratification), for example, the composition of particles anywhere in the bed is the feed composition.
2. In a uniform mixture small particles follow in the wake of the larger particles. A grouping is formed in which disturbances are transmitted between particles, even at very low concentrations. Surface area of all particles in such grouping is exposed to drag, while the gravity forces (total weight) are uniformly distributed. A mixture mean diameter and a representative slip velocity are defined accordingly.

Following assumption (1)

$$\frac{(1 - \epsilon_1)}{U_{sdi}} = \text{constant} \tag{6}$$

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where subscript i represents any component in the mixture. This ratio is constant, regardless of particle size, for any and all fractions of the nonsegregated mixture fed into the column. The mean diameter of the mixture is defined in accordance with assumption (2) (see Appendix 1):

$$d_m = \frac{\Sigma (1 - \epsilon_i)}{\Sigma \frac{(1 - \epsilon_i)}{d_i}} \tag{7}$$

The representative slip velocity of particles of mean diameter dm takes the form of Equation (2):

$$V_s = V_{so} (\epsilon_T)^n \tag{8}$$

ϵ_T is the total voidage and is obtained as

$$\epsilon_T = 1 - \Sigma(1 - \epsilon_i) = 1 - m + \Sigma\epsilon_i \tag{9}$$

where m is the number of components in the mixture.

The feed rates are related to slip velocity as usual (5):

$$\frac{\Sigma U_{sdi}}{1 - \epsilon_T} \pm \frac{U_{sc}}{\epsilon_T} = V_{so} (\epsilon_T)^n \tag{10}$$

where the plus refers to countercurrent flow and the minus to downward concurrent flow. The mean diameter d_m is calculated for specified particle sizes d_i and feed rates U_{sdi}. The terminal velocity V_{so} and the exponent n of a mixture of mean diameter d_m are obtainable from correlations of single sized particles of equivalent diameter. These values and the specified flow rates of both phases ΣU_{sdi} and U_{sc} are substituted into Equation (10) and the equation is solved for its single unknown, the total voidage ϵ_T.

In countercurrent flow, the solution usually has two roots: the restrained and the free voidage (5). A single-valued root represents flooding conditions. The equation is unsolvable above that point (16).

EXPERIMENTAL PROCEDURE

Apparatus

The fluidization and gamma ray apparatus are shown diagrammatically in Figure 1. Fluidization was carried out in a glass column 1 in. in diameter and 5 ft. long. Two glass stop cock valves of the same inner bore as the column were fitted into the column at a distance of 3 ft. Two glass 1 in. × 6 in. reducers were attached at both ends of the column and connected to copper tanks, for particle reception at the bottom and for particle feed at the top. Particle flow rates were regulated by calibrated orifices. Water flow rate was measured by rotameters. Holdup of particles in the column proper was measured locally by recording the transmitted radiation of a horizontal gamma ray beam directed through the bed from an external Co^{60} (1 μc.) source; a similar technique was employed formerly by other investigators (21, 25 to 27). The detected gamma radiation transmitted through the bed was calibrated against a measured average holdup. Average holdup was obtained from the initially known packed-bed holdup and the initial and final depths of the expanded bed. At the same time the gamma ray apparatus was moved along the expanded bed to ascertain longitudinal uniform transmission, which signified uniform distribution of particles and equal local and average holdups. The expansion of the bed and the radiation detection were repeated for four sizes of glass beads.

The transmitted radiation I at any holdup and the radiation through a column of water I_w, both expressed as the number of counts per minute, were plotted in Figure 2 as $(I_w - I)/I$ against measured holdup. The straight line formed was tested by an F test and found to be the best fit for these data.

Materials

Glass beads were fluidized by water. Table 1 lists the physical properties of the Superbrite glass beads of catalog numbers 70, 90, 140, and 160 manufactured by 3M Company. The respective mean diameters were 0.47, 0.29, 0.096, and 0.078 mm. The size distributions of these particles were determined microscopically and are illustrated in Figure 3. The particles' density was measured by liquid displacement.

TABLE 1. PHYSICAL PROPERTIES OF GLASS BEADS

Catalog No.	Arithmetic mean diameter, μ	Measured Mean standard deviation, μ	Range, μ	Density, g./cc.
70	470	31	400 to 565	2.49
90	290	18	250 to 335	2.46
140	96	9.9	71 to 121	2.35
160	78	9.6	50 to 94	2.34

Manufacturer's specifications

Catalog No.	Typical mean diameter, μ	Possible range, μ	Density, g./cc.
70	470	230 to 860	2.5 and decreases
90	290	200 to 380	slightly in the finer
140	85	74 to 107	bead sizes
160	62	40 to 85	

Procedure

Batch fluidization of the single sized glass beads of four sizes was carried out in the usual way (4). Countercurrent and concurrent downward, fluidization of mixed sized particles proceeded as follows. Water was fed into the column. The gamma ray transmission was recorded until the counting system steadied down. The feed orifices of solids were opened, and water

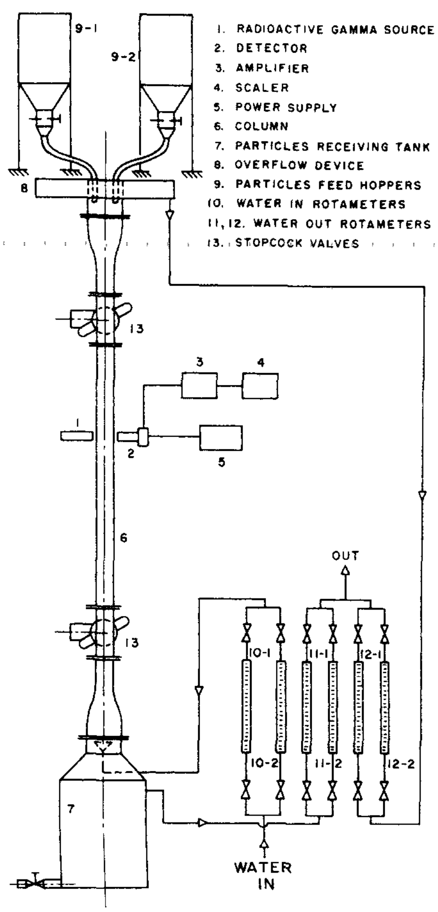

Fig. 1. Diagram of the experimental apparatus.

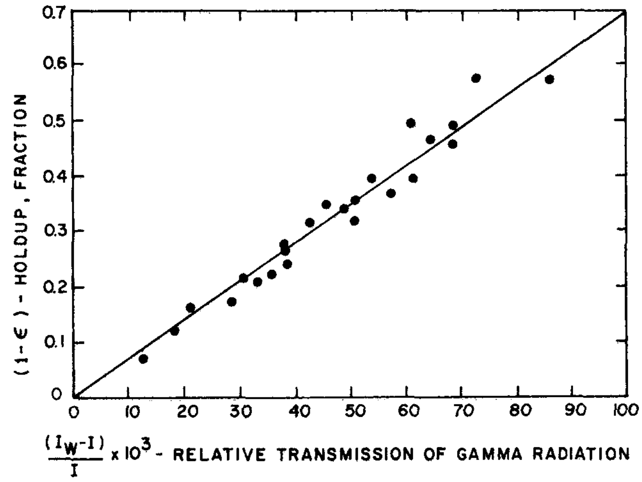

Fig. 2. Calibration of holdup versus relative transmission of gamma radiation.

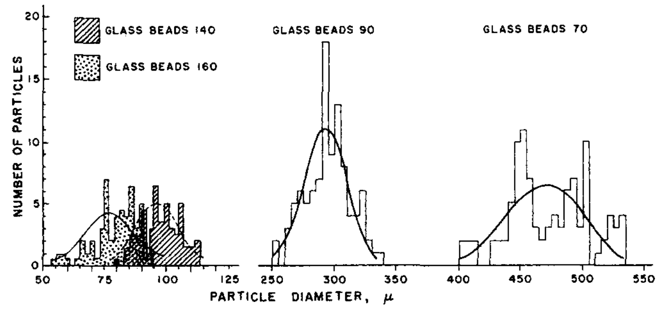

Fig. 3. Size distribution of glass beads.

flow rate adjusted. Three consecutive readings of counts at the same level were taken, 1 min. each. When the counting rates agreed within a few percent, the radiation apparatus was moved to another location along the column and the counting procedure was repeated. Holdup was calculated. Another run was initiated by increasing the water flow rate. Several runs were extended into the flooding zone. A sequence of runs was completed by closing the feed orifices and rechecking the radiation detection system through the column of water. Similar operating procedure was applied to other combinations of feed rates and ratio of particle sizes in countercurrent and concurrent flow.

RESULTS

The results of the experiments are shown in Figures 4 to 7. They can be split into two groups: (1) Batch expansion and determination of flow characteristics of single sized particles, used later as components of mixtures. (2) Measurements of holdup of binary mixtures in countercurrent and concurrent flow, and characterization of mixtures based on a mean diameter.

Batch Expansion and Flow Characteristics

The experimental expansion curves of glass beads of all four sizes were translated into the slip velocity-voidage curves and presented in the logarithmic form in Figure 4. Straight lines were obtained. Their slopes n and their slip velocities at unity voidage V_{so} were determined and correlated versus particle diameter in Figure 5.

The exponent n values decreased from 3.3 to 1.7 in the range of Reynolds numbers 0.4 to 28 and yielded a correlation of

$$n = 2.9\, N_{Re}^{-0.16} \tag{11}$$

shown in Figure 6. The Reynolds numbers in Equation (11) were based on a particle diameter and its appropriate terminal velocity. For comparison, the Richardson and Zaki correlation (*7*), illustrated in Figure 5, was interpreted in the same way.

Zenz's (*6*) original correlation of drag coefficient-Reynolds number-voidage was replotted as slip velocity-voidage curves for the system of water and glass spheres for several particle diameters (0.1 to 0.6 mm). Straight lines were obtained in the logarithmic form and their slopes were correlated versus particle diameter in Figure 5. Additional data presented for comparison in Figure 5 were Zuber's (*12*) exponent of 3.5 in the laminar region and Kramers' et al. (*28*) exponent of 2.09 in the intermediate region, obtained in a water-fluidized bed of glass spheres 0.5 mm. in diameter.

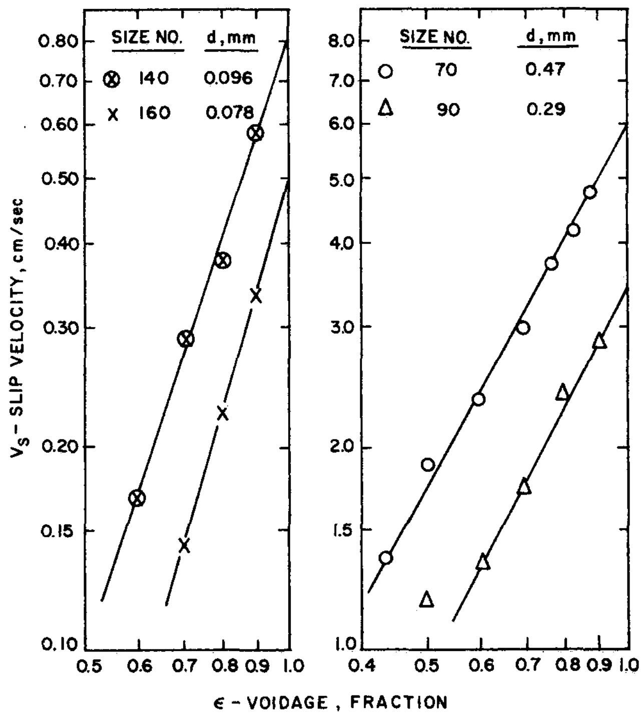

Fig. 4. Slip velocity-voidage relationship for four single sized glass beads.

Mixed Sized Particles

Binary mixtures composed of the previously analyzed single sized particles were fluidized countercurrently and concurrently. General information on the composition of these mixtures is summarized in Table 2.

Holdups of the fluidized mixtures were measured locally. Evaluated mean residence times of the mixed particles inside the 3-ft. long column proper are presented in Table 3. The calculation was based on the total flow of solids

TABLE 2. COMPONENTS OF BINARY MIXTURES

Series No.	d_1, mm.	d_2, mm.	d_1/d_2	U_{sd1}/U_{sd2}	Direction of flow
1	0.47	0.29	1.62	0.26	Countercurrent
	0.47	0.29	1.62	0.22	Countercurrent
	0.47	0.29	1.62	1.02	Countercurrent
	0.47	0.29	1.62	5.83	Countercurrent
2	0.47	0.29	1.62	0.26	Concurrent
	0.47	0.29	1.62	1.02	Concurrent
	0.47	0.29	1.62	5.83	Concurrent
3	0.29	0.096	3.02	0.91	Concurrent
4	0.47	0.028	6.05	1.02	Concurrent

and the experimentally measured holdup.

Uniform distribution of holdup was recorded along the column for each run presented. The longitudinal uniformity of holdup signified absence of stratification inside the column. Consequently, the mixtures were assumed to be uniform and therefore the previously formulated analysis for nonsegregated flow was applied. Using Equations (6) and (7), we calculated the binary mean diameter as

$$d_m = d_1 \left(\frac{a + 1}{a + d_1/d_2} \right) \tag{12}$$

where $a = U_{sd_1}/U_{sd_2}$. Values of terminal velocity V_{so} and exponent n were interpolated from the present experimental correlations (Figure 5) and substituted into Equation (1). The equation was solved for its single unknown, the total voidage ϵ_T. The results expressed as a total holdup $(1 - \epsilon_T)$ were compared with the experimentally measured holdups in Figure 7.

Some runs (not presented) were extended into the flooding region. For these runs Equation (10) was unsolvable.

DISCUSSION

The present work was intended to test a method to analyze beds of mixed particle sizes subjected to short-period fluidization. The reliability of such tests depended on the accuracy of the flow characteristics data applied to the analysis. Data from the experimental expansion curves of the single sized components were chosen and incorporated in the analysis of their mixtures. Additional use was made of the batch expansion curves in comparison with common correlations.

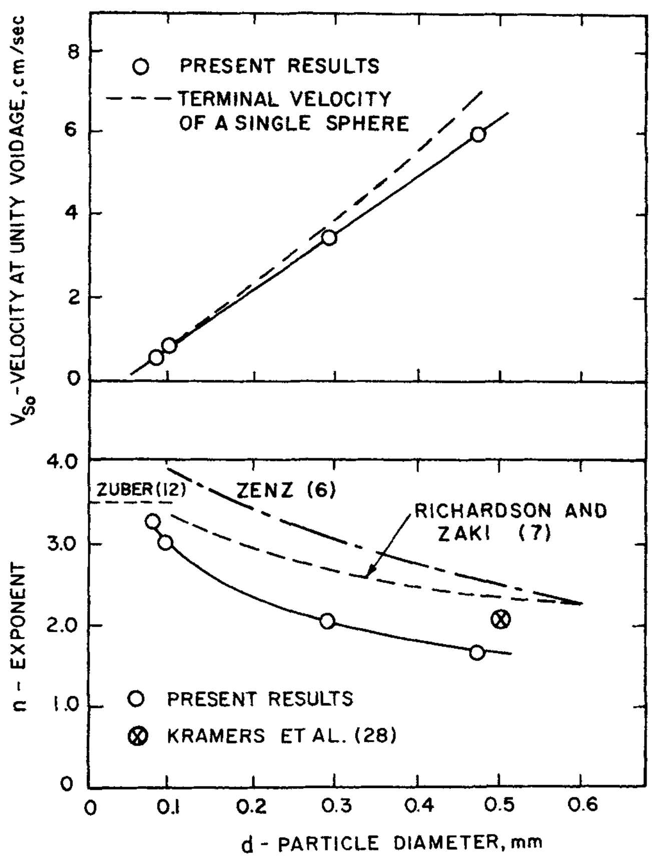

Fig. 5. Exponent n and velocity at unity voidage versus particle diameter.

Batch Expansion and Flow Characteristics

The present work extended from the laminar flow regime into the intermediate region. In this range of flows, a single slope n of any slip velocity-voidage curve for a specified particle size was determined. Practically, a single mean slope appeared satisfactory (Figure 4 and reference 28). The present experimental values of the exponent n were lower than the data of other investigators; it varied from 3.4 to 1.7 in the range of Reynolds numbers 0.4 to 28. Zenz's values (6) varied in the same range from 4.0 to 2.5. Richardson and Zaki's (7) exponents were comparable with the present results in the laminar region and approached Zenz's values in the intermediate region. Zuber's exponent value of 3.5 (12) was also comparable in the laminar region, while in the intermediate region the closest value was obtained from Kramers' et al. work (28).

The largest difference between the presented exponents was up to 50%, obtained in comparison with Zenz's data (6). However, the dependent variables in that case differed by much less. For instance, glass spheres 0.5 mm. in diameter, fluidized by water, flow at a reduced slip velocity of 0.5. Their voidage would be estimated as 0.66 if the present correlation were used ($n = 1.7$), or 0.75 if Zenz's correlation were used ($n = 2.5$). These values indicate an error of 14% in voidage due to an error of 50% in the exponent n. Zuber (12) recalculated data of other investigators using an exponent n of 3.5 for a range of Reynolds numbers from 0.07 to 58, and still obtained satisfactory agreement.

In fact, in the laminar region a reasonable agreement of

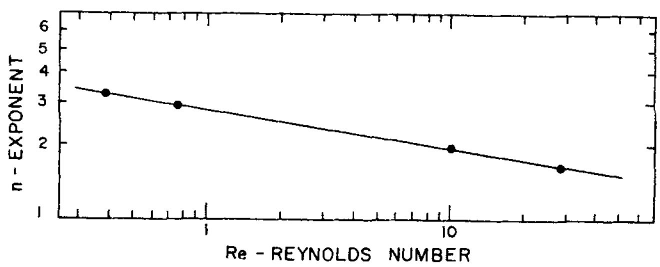

Fig. 6. Exponent n versus Reynolds number at unity voidage.

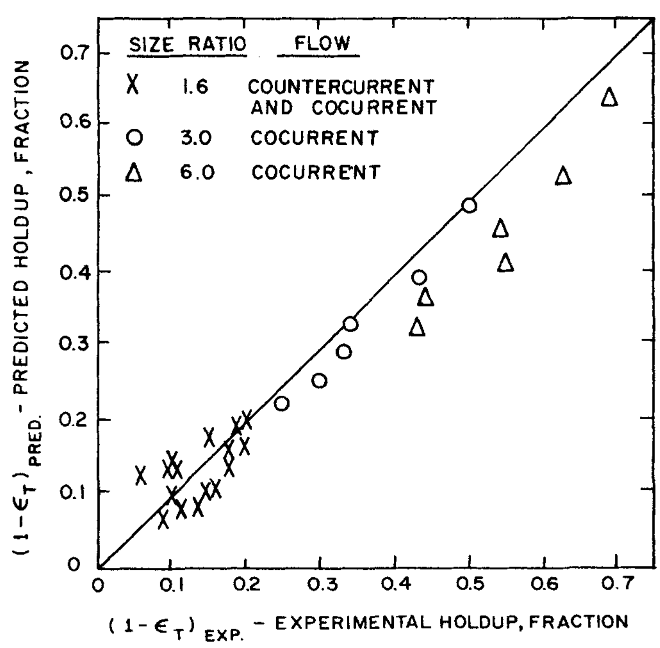

Fig. 7. Predicted holdup for flow of uniform mixtures versus experimentally measured holdup.

all data is demonstrated (Figure 5). The differences become appreciable in the intermediate region.

In general, common correlations could be used satisfactorily for practical estimates. For analytical purposes, original data are preferable, because shape and roughness of particles affect their flow characteristics.

Binary Mixtures

An analysis based on uniform flow was applied to binary mixtures of particles fluidized countercurrently and concurrently in the 3-ft. long bed. Two main experimental facts led to the assumption of nonsegregated flow: uniform longitudinal holdup and short residence times. Additional indirect evidence was deduced from sedimentation data, which indicated that segregation was prevented in concentrated (*21*) and in fast-settling suspensions (*22*). The present experimental work extended over a range of holdups from 0.08 to 0.69, while the estimated residence times did not exceed 1 min.

Binary mixtures of size ratio 1.6 and 3.0, in a holdup range of 0.08 to 0.50, in countercurrent and concurrent flow showed good agreement between the measured and predicted holdups (Figure 7). The assumption and analysis of uniform flow appeared to be justified even at the low holdups, due to the short-period fluidization.

In concurrently fluidized binary mixtures of size ratio 6.0, the measured holdups in the range of 0.43 to 0.69 were higher than the predicted values for uniform flow. The deviation was 17% at a holdup of 0.43 and decreased to 7% at a holdup of 0.69. This deviation indicated a certain degree of segregation of the mixture, either longitudinal or radial. Longitudinal segregation, that is, stratification for size ratio as high as 6.0, was expected and strongly supported by Pruden and Epstein's (*24*) analysis and Hoffman's (*4*) work. Against segregation spoke the high holdup, the measured longitudinal uniformity of holdup, and the short residence times (0.25 to 0.4 min.) inside the column (Table 3).

Neither was misevaluation of flow characteristics the origin of that deviation, because in the present case the operating line [left-hand side of Equation (10)] became negative at higher holdups.

A partial radial segregation due to an entrance effect (*29*) and stabilized by the high holdup might also be an interpretation of the deviation. In such segregation, the binary mixture is bypassed by single sized regions ruled by their individual flow characteristics.

This work did not define general criteria for uniform flow of mixtures. It attempted only to show that such flow is possible in fluidized beds. Segregation and subsequent stratification would certainly increase in longer columns, at lower holdups, at higher size ratios, and in mixtures of smaller particles.

Stratified columns would be flooded at lower velocities than predicted for a mixture because of the accumulation of the smallest particles in the upper layers. Selective flooding of stratified columns seems a possible practical application to separation of mixtures. Such a method of separation would require lower fluid velocities than those applied in elutriation.

CONCLUSIONS

Flow characteristics of single sized spheres, required for practical estimates, may be extracted from common correlations. Batch expansion curves are recommended for precise work or characterization of particles of arbitrary shapes.

Flow of mixtures of moderate size ratio in short fluidized beds is characterized by an absence of size stratification, as indicated by the uniform longitudinal holdup.

A fluidized uniform mixture may be characterized by a mean diameter and the respective flow characteristics.

Presently, general criteria for segregation of mixtures have not been defined. However, it was shown that for residence times up to 1 min., holdup down to 0.08, and size ratio up to 3.0, the binary mixtures did not segregate.

Investigation of dynamic stratification would contribute to improved design and control of nonuniform systems.

TABLE 3. RANGE OF RESIDENCE TIMES OF SOLIDS IN THE COLUMN PROPER

Series No.	1	2	3	4
Range of residence Times, min.	0.2 to 1.0	0.15 to 0.7	0.2 to 0.45	0.25 to 0.4

NOTATION

a = feed ratio of a binary mixture
A_i = surface area of particles of size i per unit volume
d = particle diameter
F_A = friction forces
I = transmitted gamma radiation through solids, counts/min.
I_w = transmitted gamma radiation through water, counts/min.
k = flow regime index, Equation (5)
m = number of components in a mixture
n = exponent, Equation (2)
n_i = number of particles of size i per unit volume
U_{sc} = superficial velocity of continuous phase
U_{sd} = superficial velocity of dispersed phase
V_i = volume of particles of size i per unit volume
V_s = slip velocity
V_{so}, V_T = velocity at unity voidage
X_i = weight fraction of solids, Equation (4)

Greek Letters

ϵ = voidage, fraction
$(1 - \epsilon)$ = holdup, volume fraction of solids
ρ = density
ρ_B = bulk density, Equation (5)

Subscripts

c = continuous phase
d = dispersed phase
i = each component in a mixture
m = mean
T = total

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APPENDIX 1: DEFINITION OF THE MEAN DIAMETER OF A UNIFORM MIXTURE

The friction forces on the total surface (ΣA_i), exposed by the particles in a mixture, equilibrate with the gravity forces, acting on the total volume (ΣV_i) of all the particles of the same density.

$$F_A \Sigma A_i = (\rho_d - \rho_c)\, g \Sigma V_i \tag{1A}$$

The number of particles of any size i per unit volume is

$$n_i = \frac{1 - \epsilon_i}{\dfrac{\pi}{6} d_i^3} \tag{2A}$$

while

$$A_i = n_i \cdot \frac{\pi}{4} d_i^2 \tag{3A}$$

and

$$V_i = n_i \cdot \frac{\pi}{6} d_i^3 \tag{4A}$$

Substitution of Equations (2A), (3A), and (4A) into Equation (1A) yields

$$\frac{3}{2} \frac{F_A}{(\rho_d - \rho_c)} = \frac{\Sigma(1 - \epsilon_i)}{\Sigma \dfrac{1 - \epsilon_i}{d_i}} \tag{5A}$$

The definition of the mean diameter d_m of a mixture assumes that the active forces are the same.

$$\frac{\Pi}{4} d_m^2 \cdot F_A = \frac{\Pi}{6} d_m^3 (\rho_d - \rho_c)\, g \tag{6A}$$

$$d_m = \frac{3}{2} \frac{F_A}{(\rho_d - \rho_c)\, g} \tag{7A}$$

Comparison of Equations (5A) and (7A) yields the definition of the mean diameter d_m as expressed in Equation (7).

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Stability of Nonlinear Systems Containing Time Delays and/or Sampling Operations

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In recent years the stability of nonlinear systems has been a topic of considerable interest. Relatively little attention, however, has been directed toward nonlinear time-delay systems, that is, systems whose dynamic behavior is described by a set of differential-difference equations. Although the fundamental stability theorems of Liapunov's second method have been extended to include time-delay systems (*3, 6, 7, 10*), difficulties in implementation have severely limited applications. Most applications have been concerned with the local stability of nonlinear systems (*8, 10, 16*), linear systems (*10, 16, 18, 19*), or linear systems containing simple nonlinear elements (*9, 23*). Quantitative estimates of the region of stability (or asymptotic stability) have not previously been reported for nonlinear time-delay systems. Such estimates would provide useful information concerning the range of disturbances for which a stable

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